Mid-Western University

Examinations Management Office

Chance Examinations -2081

Bachelor level/ B.Sc. / 8th Semester F.M.: 100

Time: 3hrs P.M.: 50

Subject: Advance Calculus I (MATH 483)

Candidates are required to give their answer in their own words as far as Practicable. The figures in the margin indicate full marks.

Group 'A' **[$6 \times (2+2) = 24$]**

1. a) For each $\vec{a}, \vec{b} \in \mathbf{R}^n$, show that, $|\vec{a}+\vec{b}| = |\vec{a}|^2 + 2\,\vec{a}.\vec{b} + |\vec{b}|$.

   b) For each $\vec{a}, \vec{b} \in \mathbf{R}^n$, show that $\left||\vec{a}| - |\vec{b}|\right| \le |\vec{a}-\vec{b}|$.

2. a) Let $f(x) = \begin{cases} x \text{ if } x \text{ is rational} \\ 0 \text{ if } x \text{ is irrational} \end{cases}$.

   Show that f is continuous at $x = 0$ and nowhere else.

   b) If $u = f(2x - y^2, x\sin 3y, x^4)$, find $\partial_y u$.

3. a) If $f(x) = x\cos\frac{1}{x}$, $for\ x \neq 0\ and\ f(0) = 0$ show that f is differentiable $\forall x \in \mathbf{R}$.

   b) Compute differential df if $f(x,y) = e^{4x^2y - y^2}$.

4. a) Find the tangent plane to the surface $z = x^2 - y^3$ at $(2,-1,5)$.

   b) Can the equation $x^2 - 4x + 2y^2 - yz = 1$ be solved uniquely for y in terms of x and z near $(2,-1,3)$?

5. a) If $(u,v) = f(x,y) = (x-2y, 2x-y)$, calculate the inverse transformation $(x,y) = f^{-1}(u,v)$.

   b) is the function $f(x) = \begin{cases} \sqrt{1-x^2}, & if\ x\ is\ rational \\ 1-x, & if\ x\ is\ irrational \end{cases}$

   is integrable on [0,1]

6. a) Find the region of integral for the other iterated integral

   $\int_0^1 \int_{x^2}^{\frac{1}{3}} f(x,y)\,dy\,dx.$

   b) Define improper integrals with examples.

Group 'B' **[$13 \times 4 = 52$]**

7. Let $\{x_n\}$ and $\{y_n\}$ be sequence in R such that $x_n \to a$ and $y_n \to b$. Show that $x_n + y_n \to a + b$ and $x_n y_n \to ab$.

8. "A sequence $\{x_n\}$ in $\mathbb{R}^n$ is convergent if and only if it is Cauchy". Prove it.

9. If $f(x, y) = \frac{xy(x^2 - y^2}{x^2+y^2}$ if $(x, y) \neq (0, 0)$, and $f(0, 0) = 0$.

   Verify that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

10. If $u = f(xz, yz)$,then show that $x\partial_x u + y\partial_y u = z\partial_z u$.

11. Define directional derivatives of f at $\vec{a}$ in the direction $\vec{u}$. If f is differentiable at $\vec{a}$ then show that the directional derivatives of f at $\vec{a}$ exists and $\partial_u f(\vec{a}) = \nabla f(\vec{a}).\vec{u}$.

    OR

    If $u = F(x + g(y))$, then prove that $u_x u_{xy} = u_y u_{xx}$.

12. Find the maximum and minimum value of the function $f(x, y) = xy\,(12 - 3x - 4y)$.

13. Find third order Taylor polynomial of $f(x,y) = e^{x^2+y}$ about $(x,y) = (0,0)$.

14. Find an equation for the tangent plane to the surface

$x = \frac{1}{u+v}, \quad y = -(u + e^v), z = u^3$ at the point $(1, -2, 1)$.

15. For the transformations $(u, v) = f(x, y)$ compute Jacobian det Df, where $u = x^2 + 2xy + y^2, v = 2x + 2y$.

16. If f is continuous on [a, b], then f is integrable on [a, b].

17. Evaluate the iterated integral $\int_0^1 \int_{\sqrt{x}}^1 \cos(y^3 + 1)\, dx\, dy$.

18. For the regions $S \subset \mathbb{R}^2$, express the double integral $\iint_S f\, dA$ in terms of iterated integrals in two ways, where S is the region between the parabolas $y = x^2$ and $y = 6 - 4x - x^2$

OR

Evaluate $\int_1^2 \int_{1/x}^1 y e^{xy} dy\, dx$.

19. Determine whether the improper integral converges;
$\int_0^\infty x^{-1/3}(1 - x)^{-2} dx$.

Group 'C' [$4 \times 6 = 24$]

20. Define monotone sequence in $\mathbb{R}$. Prove that every bounded sequence in $\mathbb{R}$ has convergent subsequence.

21. Let S be the circle formed by intersecting the plane $x + z = 1$ with the sphere $x^2 + y^2 + z^2 = 1$. Find a parametrization of S and parametric equations for the tangent line at $\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$.

22. If $u = f(x, y), x = s^2 - t^2, y = 2st$ then show that
$\partial_s{}^2 u + \partial_t{}^2 u = 4s^2 - t^2)(\partial_x{}^2 f + \partial_y{}^2 f)$.

OR

Suppose u is a function of (x, y) in some open set in $\mathbb{R}^2$. If (x, y) is related to (r, θ) by $x = r\cos\theta, y = r\sin\theta$. Then $f_{xx} + f_{yy} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$.

23. Find the volume of the region above the triangle in the xy-plane with vertices (0,0), (1,0) and (0,1), and below the surface
$z = 6xy(1 - x - y)$

THE END