Mid-West University

Examinations Management Office

Surkhet,Nepal

End Semester Examinations -2078

Bachelor level/ B.Sc/7th Semester — Full Marks : 100

Time: 3 hrs — Pass Marks : 50

Subject : Real Analysis II (MATH 471)

Candidates are required to give their answers as far as practicable. The figures in the margin indicate full marks.

Group A

Attempt All the Questions **[6 x (2 x 2)=24]**

1. a. Prove that $\forall x \neq 0, \frac{d}{dx} ln|x| = \frac{1}{x}$.

   b. Suppose f is differentiable function in an interval I and $\forall\, x \in$ I, $f'(x) = 0$ then f is constant on I.

2. a. Find the 4th Taylor polynomial of the function $f(x) = \cos x$ about 0.

   b. Show that a constant function $f(x) = c$ is integrable over $[a, b]$ and $\int_a^b f = c(b-a)$.

3. a. For all partition $\mathcal{P}$ of $[a, b]$ prove that $\underline{S}(f, \mathcal{P}) \leq \bar{S}(f, \mathcal{P})$.

   b. Define Upper and Lower Riemann integral.

4. a. If f is integrable over $[a, b]$ and $c \in \mathbb{R}$, cf is integrable and
$$\int_a^b cf = c\int_a^b f$$

   b. Define alternating series.Prove that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

5. a. Define the terms

   i. Analytic function **ii.** Taylor Polynomial for a functions

   b.Find the sum if the series $\frac{1}{4} - \frac{1}{16} + \frac{1}{64} - \frac{1}{256} + \ldots\ldots\ldots$ converges.

6. a. If $\sum_{k=0}^{\infty} f_k = f$ uniformly on a set S , then $\|f_n\| \to 0$.

   b. Find the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k2^k}$.

Group B

Attempt Any Thirteen Questions **[13 x 4 = 52]**

7. Suppose f and g differentiable at x then $\frac{f}{g}$ is differentiable at x and
$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}, where\ g(x) \neq 0.$

8. a. Suppose $f'(x) > 0$ at an interior point x_0 of D (f). Then $\exists\, \delta > 0$ such that a) $\forall\, x \in (x_0 - \delta, x_0), f(x) < f(x_0)$ and

   b. $\forall\, x \in (x_0,\ x_{0+\partial}), f(x) > f(x_0)$.

9. Find the 4^{th} Taylor Polynomial of the function $f(x) = 3 + 5x^2 - 4x^3 + x^4$ about $x = 1$.

10. Every differentiable function is continuous. Is converse also true ? justify with example.

OR

Show that the function $f(x) = \begin{cases} x\ sinx & \text{for} x \neq 0 \\ 0 & \text{for} x = 0 \end{cases}$ is continuous at 0 but not differentiable at 0.

11. If f: [c, d] $\to \mathbb{R}$ is integrable on [a, b], then $\forall x \in (a, b)$, $\int_a^b f = \int_a^x f + \int_x^b f$.

12. If f is continuous on the interval [a , b] then f is integrable on [a , b].

13. Suppose f and g are differentiable on [a , b] and their derivatives f' and g' are integrable on [a , b]. Then both fg' and gf' are integrable on [a , b], and $\int_a^b fg' + \int_a^b gf' = f(b)g(b) - f(a)g(a)$.

14. Suppose $\sum a_n$ and $\sum b_n$ are non negative series and $\exists n_o \in \mathbb{N}$ such that $n \geq n_o \Longrightarrow a_n \leq b_n$, then

    a) if $\sum b_n$ converges , then so does $\sum a_n$

    b) if $\sum a_n$ diverges , then so does $\sum b_n$.

15. If $f_n \to f$ uniformly on a closed interval [a , b] and if each f_n is integrable on [a , b] , then f is integrable on [a, b] and $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$.

OR

16. Suppose $\{f_n\}$ is a sequence of functions in $\mathcal{F}(S, \mathbb{R})$ then $\{f_n\}$ converges uniformly to some f in $\mathcal{F}(S, \mathbb{R})$ if and only if
$\forall \varepsilon > 0, \exists n_o \in \mathbb{N} : m, n \geq n_o \Longrightarrow f_n - f_m$ is bounded and $\|f_n - f_m\| < \varepsilon$.
17. If $f : [a, b] \to \mathbb{R}$ is bounded and $a < c < b$ then $\overline{\int_a^b} f = \overline{\int_a} f + \overline{\int_c^b} f$.
18. If f is integrable on [a , b] then so does $|f|$ and
$\left|\int_a^b f\right| \leq \int_a^b |f| \leq M(b - a)$ when $|f| \leq M$ on [a , b].

19. 19. Given any sequence $\{x_n\}$ of positive real numbers , then
$\underline{\lim}_{n\to\infty} \frac{x_{n+1}}{x_n} \leq \underline{\lim}_{n\to\infty} \sqrt[n]{x_n} \leq \overline{\lim}_{n\to\infty} \sqrt[n]{x_n} \leq \overline{\lim}_{n\to\infty} \frac{x_{n+1}}{x_n}$.
20. Define absolutely converge and conditionally converge. Consider a series $\sum b_n$ of real numbers then $\sum b_n$ converges absolutely if and only if both $\sum b_n^+$ and $\sum b_n^-$ converge.

Group C

Attempt Any Four Questions **[4 x 6 = 24]**

21. Define differentiable function. If f and g are differential at x then fg is differential at x and $fg(x) = f(x)g'(x) + g(x)f'(x)$.
22. Define upper and lower Riemann sum. If $f(x)$ = C constant function over the interval [a, b] , then $\int_a^b f = C(b - a)$.
23. If there exit a sequence $\{\mathcal{P}_n\}$ and $\{Q_n\}$ of partition of [a , b] such that $\underline{S}(f, \mathcal{P}_n) \to L$ and $\overline{S}(f, Q_n) \to L$ then f is integrable on [a , b] and $\int_a^b f$. Use this result to find the integral of $f(x) = x^2$ on [0 , 1].

OR

Define derived series. If a function f is representable as a power series with non-zero radius of convergence , then f is differentiable at every point in the interior of its interval of convergence ; moreover , its deriveseries is its derivative ;that is $f(x) = \sum_{k=1}^{\infty} a_k(x - c)^k$ with interval of convergence I , then at every point in the interior of I , $f'(x) = \sum_{k=1}^{\infty} a_k k(x - c)^{k-1}$.
24. Suppose $\{f_n\}$ converges uniformly to f on a set S - $\{x_0\}$ for some x_0 in S. If each f_n has a (finite) limit as $x \to x_0$, then so does f ,and we can interchange the limit. More precisely ,if $\forall n \in \mathbb{N}, \lim_{x\to x_o} f(x)$ exists and

$$\lim_{x\to x_o} (\lim_{n\to\infty} f_n(x)) = \lim_{n\to\infty} (\lim_{x\to x_o} f_n(x)).$$

THE END