Mid-Western University

Examinations Management Office

Chane Examinations -2081

Bachelor level/ B.Sc /7th Semester — Full Marks : 100

Time: 3 hrs — Pass Marks : 50

Subject : Real Analysis II (MATH471)

Candidates are required to give their answers as far as practicable. The figures in the margin indicate full marks.

Group A

Attempt All the Questions **[6 x (2 x 2)=24]**

1. a) By using the definition , show that the function $f(x) = \sqrt{x}$ is differential on (0, +∞) and for all x_o in (0, +∞) $f'(x) = \frac{1}{2\sqrt{x}}$.

   b. Suppose f is differentiable function in an interval I and $\forall x \in I, f'(x) = 0$ then f is constant on I

2. Find the limit using L' Hospital's rule $\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$

   b. If $A \subseteq \mathbb{R}$ is bounded and $x \in \mathbb{R}$ then Sup ($x + A$) = x + Sup A

3. a. For all partition $\mathcal{P}$ of [a , b] prove that $\underline{S}(f, \mathcal{P}) \le \bar{S}(f, \mathcal{P})$

   b) Define Upper and Lower Riemann integral.

4. a)Suppose $f: [a, b] \to R$ is bounded and nonnegative on [a,b], then prove that $\int_{\underline{a}}^{b} f \ge 0$.

   b) If the series $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

5. a) Define a power series with suitable example.

   b. If $\{f_n\}$ is a sequence of function in B (S) converging uniformly on S to real – valued function f , then $f \in$ B (S)

6. a) If $\sum_{k=0}^{\infty} f_k = f$ uniformly on a set S , then $\|f_n\| \to 0$.

   b) Find the radius of convergence of the power series $\sum_{k=1}^{\infty} \frac{(x-3)^k}{k2^k}$.

Group B **[13 x 4 = 52]**

7. Differentiability implies continuity but converse may not be true. Justify your answer with example.

8. State Mean- Value Theorem. Using Mean- Value Theorem for any differential function in the interval I $f'(x) = 0 \ \forall x \in I$ .then f is constant .

9. Find the 4^{th} Taylor Polynomial of the function $f(x) = 3 + 5x^2 - 4x^3 + x^4$ about 1

10. Consider the characteristic function defined by $(x) = \begin{cases} 1 & if\ 3 < x < 6 \\ 0 & otherwise \end{cases}$. Prove that f is integrable on [0,10] and find $\int_0^{10} f$.

11. If f: [c, d] → ℝ is integrable on [a, b], then $\forall x \in (a, b)$, $\int_a^b f = \int_a^x f + \int_x^b f$.

    OR

    If f is continuous on the interval [a , b] then f is integrable on [a , b].

12. If f monotone on [a, b,] ,then f is integrable on [a, b,]

13. State and prove First Fundamental Theorem of Calculus.

14. If $f_n \to f$ uniformly on a closed interval [a , b] and if each f_n is integrable on [a , b] , then f is integrable on [a, b] and $\int_a^b f = \lim_{n\to\infty} . \int_a^b f_n$.

    OR

    If $f : [a, b] \to \mathbb{R}$ is bounded and $a < c < b$ then $\overline{\int_a^b} f = \overline{\int_a^c} f + \overline{\int_a^b} f$.

15. Let $\sum a_n$ be non-negative series

    a) if $\exists\ 0 < r < 1$ such that $\sqrt[n]{a_n} \le r$ for all but finitely many n ,then $\sum a_n$ converges.

    b) if $\sqrt[n]{a_n} \ge 1$ for finitely many n , then $\sum a_n$ diverges

16. Define the sequence $\{a_n^+\}$ and $\{a_n^-\}$. A series $\sum a_n$ converges absolutely if and only if both $\sum a_n^+$ and $\sum a_n^-$ converges.

17. If f is integrable on [a , b] then so does $|f|$ and $\left|\int_a^b f\right| \le \int_a^b |f| \le M(b-a)$ when $|f| \le M$ on [a , b].

18. Given any sequence $\{x_n\}$ of positive real numbers , then $\underline{\lim}_{n\to\infty} \frac{x_{n+1}}{x_n} \le \underline{\lim}_{n\to\infty} \sqrt[n]{x_n} \le \overline{\lim}_{n\to\infty} \sqrt[n]{x_n} \le \overline{\lim}_{n\to\infty} \frac{x_{n+1}}{x_n}$.

19. Define absolutely converge and conditionally converge. Consider a series $\sum b_n$ of real numbers then $\sum b_n$ converges absolutely if and only if both $\sum b_n^+$ and $\sum b_n^-$ converge.

Group C **[4 x 6 = 24]**

20. Define differential function. Suppose f is differential at an interior point x_0 of its domain, and g is differential at $f(x_0)$, an interior point of its domain. Then gof is differential at x_0 and

$$(gof)'(x_0) = g'of(x_0).f'(x_0) = g'(f(x)).f'(x_0)$$

21. A bounded function $f: [a, b] \to \mathbb{R}$ is integrable over $[a, b] \Leftrightarrow \forall \varepsilon > 0, \exists$ partition $\mathcal{P}$ of $[a, b]$ s. t.

$\overline{S}(f, \mathcal{P}) - \underline{S}(f, \mathcal{P}) < \varepsilon$

22. Let p be a fixed real number. The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n+6}{\sqrt{n^3+2}}$.

OR

Define derived series. If a function f is representable as a power series with non-zero radius of convergence, then f is differentiable at every point in the interior of its interval of convergence; moreover, its deriveseries is its derivative; that is $f(x) = \sum_{k=1}^{\infty} a_k(x - c)^k$ with interval of convergence I, then at every point in the interior of I, $f'(x) = \sum_{k=1}^{\infty} a_k k(x - c)^{k-1}$.

23. Suppose $\{f_n\}$ converges uniformly to f on a set S - $\{x_0\}$ for some x_0 in S. If each f_n has a (finite) limit as $x \to x_0$, then so does f, and we can interchange the limit. More precisely, if $\forall n \in \mathbb{N}$, $\lim_{x \to x_0} f(x)$ exists and

$$\lim_{x \to x_0} (\lim_{n \to \infty} f_n(x)) = \lim_{n \to \infty} (\lim_{x \to x_0} f_n(x)).$$

THE END